SOLUTIONS TO MATHEMATICS PROBLEMS

1. \((1 - y^2) + 2xy = 0\)
   \((1 - y^2) = -2xy\)
   \(-\frac{1 - y^2}{2y} = -x\)

   Substituting the given solutions shows \((1, i)\) is the correct answer.

2. On the \(x\)-\(y\) plane, choose two foci at \(F(c,0)\) and \(F'(-c,0)\) and denote the constant distance as \(2a\). Then,

   \[\sqrt{(x - c)^2 + (y - 0)^2} - \sqrt{(x + c)^2 + (y - 0)^2} = 2a\]

   using the distance formula and squaring both sides this can be rewritten as

   \[\left(\sqrt{(x - c)^2 + (y - 0)^2} - \sqrt{(x + c)^2 + (y - 0)^2}\right)^2 = (2a)^2\]

   \[
   \frac{x^2 - y^2}{a^2} - \frac{x^2 + y^2}{c^2 - a^2} = 1
   \]

   let \(b^2 = c^2 - a^2\), then

   \[
   \frac{x^2 - y^2}{a^2} - \frac{y^2}{b^2} = 1
   \]

   which is the equation for a hyperbola.

3. \(v_p = \frac{1}{3}v_j\), \(d = v \cdot t\)

   \(v_p \cdot t_p = 3000\), \(v_j \cdot t_j = 3000\), \(t_p = t_j + 10\)

   \[
   \left(\frac{1}{3}v_j\right)(t_j + 10) = 3000
   \]

   substituting in \(t_j = 3000 / v_j\)

   \[
   1000 + \frac{10}{3}v_j = 3000
   \]

   \[
   \therefore \, v_j = 600 \text{ mph}
   \]

4. \(x^2 + y^2 - 2x - 4y - 17 = 0\)

   complete the square

   \((x^2 - 2x + 1) + (y^2 - 4y + 4) = 17 + 1 + 4\)

   \((x - 1)^2 + (y - 2)^2 = 22\)

   Therefore, the center is \((1, 2)\).

5. \[
\frac{a^4 + b^4}{a^2 + b^2} = \frac{a^4 + 2a^2b^2 + b^4 - 2a^2b^2}{a^2 + b^2}
\]

   \[
   = \frac{(a^2 + b^2)^2 - 2a^2b^2}{a^2 + b^2}
   \]

   \[
   = a^2 + b^2 - \frac{2a^2b^2}{a^2 + b^2}
   \]
6. \[ \frac{3+2i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{9+12i+4i^2}{9-4i^2} = \frac{5+12i}{13} \]

7. Using Cramer's Rule:

\[
\begin{vmatrix}
5 & -4 & 2 \\
-3 & 4 & 0 \\
1 & 0 & 4 \\
\end{vmatrix} = 5 \cdot (16) + 4 \cdot (-12) + 2 \cdot (-4) = 24
\]

\[
X = \frac{\begin{vmatrix}
6 & 4 & 0 \\
-3 & 6 & 0 \\
1 & 6 & 4 \\
\end{vmatrix}}{24} = \frac{48}{24} = 2,
Y = \frac{\begin{vmatrix}
5 & 0 & 2 \\
-3 & 6 & 0 \\
1 & 6 & 4 \\
\end{vmatrix}}{24} = \frac{72}{24} = 3,
Z = \frac{\begin{vmatrix}
5 & 4 & 0 \\
-3 & 4 & 6 \\
1 & 0 & 6 \\
\end{vmatrix}}{24} = \frac{24}{24} = 1
\]

Then, \( x = \frac{48}{24} = 2, \ y = \frac{72}{24} = 3, \ z = \frac{24}{24} = 1 \)

8. Logarithms of base numbers are used to simplify complicated numerical computations involving products, quotients, and powers of real numbers. Base 10 is commonly employed as it is well suited to decimal form. The natural logarithm of base e, which governs many natural phenomena is the inverse function of the natural exponential function. Thus, \( y = \ln(x) \) if and only if \( x = e^y \).

9. area in square feet = circumference in feet

\[ \pi r^2 = 2\pi r \]

\[ r^2 = 2r \]

\[ r = 2 \]

10. A circle with center, C, and radius \( r > 0 \) consists of all points in the plane that are \( r \) units from C. A point \( P(x,y) \) is on the circle if and only if \( d(C,P) = r \), or by the distance formula,

\[ d(C(h,k), P(x,y)) = r \]

\[ \sqrt{(x-h)^2 + (y-k)^2} = r \]

\[ (x-h)^2 + (y-k)^2 = r^2 \]

which is the equation of the circle of radius \( r \) and center \((h,k)\).

11. \( l = 2h \)

\( w = l - 10 \)

\( A = 2lw + 2lh + 2hw = 10wh \)

Substituting in for \( h \) and \( w \) in terms of \( l \) from the previous equations,

\[ 2l(l-10) + 2l(l/2) + 2(l/2)(l-10) = 10(l-10)(l/2) \]

\[ 20l = l^2 \]

\[ \therefore l = 20, \ w = 10, \ h = 10. \]
12. \[ Ax^2 + Bx + C = 0 \]
\[ x^2 + \frac{Bx}{A} + \frac{C}{A} = 0 \]
\[ x^2 + \frac{Bx}{A} = -\frac{C}{A} \]
\[ x^2 + \frac{Bx}{A} + \frac{B^2}{4A^2} = \frac{B^2}{4A^2} - \frac{C}{A} \]
\[ \left( x + \frac{B}{2A} \right)^2 = \frac{B^2 - 4AC}{4A^2} \]
\[ x + \frac{B}{2A} = \pm \frac{\sqrt{B^2 - 4AC}}{2A} \]
\[ \therefore x = -\frac{B \pm \sqrt{B^2 - 4AC}}{2A} \]

13. A sphere encloses the maximum volume with the minimum surface area. This is verifiable using appropriate formulas for various geometric shapes.

14. Graphing these points shows this to be a parabola. Its equation is \( y = -x^2 + 4 \).

15. a. For a triangle of base, \( b \), and height \( h \):
\[ y = mx + b = \frac{h}{b} x + h \]
\[ A = \int_{b}^{0} y \, dx = \int_{b}^{0} \left( \frac{h}{b} x + h \right) \, dx \]
\[ A = \frac{h}{2b} x^2 + hx \bigg|_{b}^{0} \]
\[ A = \frac{1}{2} bh \]

b. An incremental sector of a circle can be approximated as a triangle of base \( Rd\theta \) and height \( R \). The area of a triangle is \( 1/2bh \) or \( 1/2R^2d\theta \). Integrating this over the range \( 0 < \theta < 2\pi \) yields:
\[ A = \frac{1}{2} R^2 \theta \]
\[ A = \frac{1}{2} R^2 (2\pi - 0) \]
\[ A = \pi R^2 \]
c. Using polar coordinates, the differential volume of a cone of base radius, \(R\), and height, \(h\) is:

\[
V = \iiint_0^h r^2 \sin \theta \, dr \, d\theta \, dz
\]

\[
= \int_0^h \left( h - \frac{h \cdot R}{R} \right) r^2 \sin \theta \, r \, dr
\]

\[
= 2\pi \left( \frac{hr^2}{2} - \frac{hr^2}{3R} \right)
\]

\[
= \frac{1}{3} \pi R^2 h
\]

d. Find the volume in the first quadrant using a triple integral and multiply by four. The integration becomes:

\[
0 < z < h
\]

\[
0 < x < \frac{w - wz}{2h}
\]

\[
0 < y < \frac{w - wz}{2h}
\]

\[
V = 4 \cdot \iiint_0^h \frac{w - wz}{2h} \, dx \, dy \, dz
\]

\[
= 4 \int_0^h \left( \frac{w - wz}{2h} \right) \left( \frac{w - wz}{2h} \right) dz
\]

\[
= \int_0^h \left( \frac{w^2 - 2 w^2 z}{h} - \frac{w^2 z}{h^2} \right) dz = \frac{w^2 - w^2 z}{h} - \frac{w^2 z^2}{3h^2}
\]

\[
V = \frac{1}{3} w^2 h
\]

16. a.

\[
A = \int_0^1 (x^2 - x^3) \, dx
\]

\[
A = \frac{x^3}{3} - \frac{x^4}{4} \bigg|_0^1 = \frac{1}{12}
\]

b.

\[
A = \int_0^1 (x - x^2) \, dx
\]

\[
A = \frac{x^2}{2} - \frac{x^3}{3} \bigg|_0^1 = \frac{1}{6}
\]
17. To find the area, break the integral up into three regions:

\[
A = \int_1^4 y \, dx + \int_4^6 y \, dx + \int_6^8 y \, dx = \int_{-3}^{1} \left[0 - (x^2 + x - 6)\right] \, dx + \int_{-3}^{1} \left[0 - (x^2 + x - 6)\right] \, dx + \int_{1}^{2} 0 \, dx
\]

\[
A = \frac{49}{3}
\]

18. The differential volume can be approximated as a cylinder with a differential volume \(dV = \pi R^2 H\) or \(dV = \pi y^2 \, dx\):

\[
V = \int_{1}^{2} \pi y^2 \, dx = \int_{1}^{2} \pi \left(\frac{1}{x}\right)^2 \, dx = -\frac{\pi}{x}\bigg|_{1}^{2} = \pi
\]

19. \(A = \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} r^2 \, dr \, d\theta\)

\[
A = \int_{0}^{\frac{\pi}{2}} \frac{1}{2} r^2 \bigg|_{0}^{\frac{\pi}{2}} \, d\theta = \int_{0}^{\frac{\pi}{2}} \frac{3}{2} \, d\theta
\]

\[
A = \frac{3\pi}{2}
\]

20. a. \(\int x \sin x \, dx\)

\[
u \cdot v = \int v \, du
\]

\[
u = x \quad du = dx \quad dv = \sin x \, dx \quad v = -\cos x
\]

\[
= -x \cos x + \int \cos x \, dx
\]

\[
= -x \cos x + \sin x + c
\]

b. \(\int x(x^2 - 4)^{1/2} \, dx\)

\[
= \frac{1}{3} (x^2 - 4)^{3/2} + c
\]
c. \[\int \frac{e^x - 3}{x^2} \, dx\]
   \[= \int \frac{e^x}{x^2} \, dx - \int \frac{3}{x^2} \, dx\]
   \[= -\frac{e^x}{x} + \frac{3}{2x^2} + c\]

d. \[\int \frac{e^x - 3}{x^2} \, dx\]
   \[= \int \frac{e^x}{x^2} \, dx - \int \frac{3}{x^2} \, dx\]
   \[= -\frac{e^x}{x} + \frac{3}{2x^2} + c\]

e. \[\int (x \sin^2 x + x^3) \, dx\]
   Using the substitution, \[\sin^2 x = \frac{1 - \cos 2x}{2}\]
   \[\int \left(\frac{x}{2} - \frac{x \cos 2x}{2} + x^3\right) \, dx\]
   \[u = x \quad du = dx \quad dv = \frac{\cos 2x}{2} \, dx \quad v = \frac{\sin 2x}{4}\]
   \[u \cdot v - \int v du\]
   \[= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \int \frac{\sin 2x}{4} \, dx + \frac{x^4}{4}\]
   \[= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + \frac{x^4}{4} + c\]

f. \[\int \sec u \tan u \, du\]
   \[= \int \frac{1}{\cos u} \cdot \sin u \, du\]
   \[= \int \frac{\sin u}{\cos^2 u} \, du\]
   \[= \frac{1}{\cos u} + c\]
   \[= \sec u + c\]

g. \[\int xe^x \, dx\]
   \[u = x \quad du = dx \quad dv = e^x \, dx \quad v = e^x\]
   \[u \cdot v - \int v du\]
   \[= xe^x - \int e^x \, dx\]
   \[= xe^x - e^x + c\]
   \[= e^x(x - 1) + c\]
h. $\int (y + 3)(y + 1)\,dy$
  \[= \int (y^2 + 4y + 3)\,dy\]
  \[= \frac{y^3}{3} + 2y^2 + 3y + c\]

i. $\int_0^{\pi/2} \int_0^{\pi/2} r \sin \theta \,d\theta \,d\theta \,dr$
  \[= \int_0^{\pi/2} \int_0^{\pi/2} \frac{\pi}{2} r \sin \theta \,d\theta \,dr\]
  \[= \int_0^{\pi/2} \left[ -\frac{\pi}{2} r \cos \theta \right]_0^{\pi/2} \,dr = \int_0^{\pi/2} \frac{\pi}{2} r^2 \,dr\]
  \[= \frac{\pi}{4} r^4\bigg|_0^R = \frac{\pi R^2}{4}\]

j. $\int (2x + 1)\,dx$
  \[= x^2 + x + c\]

21. a. $\frac{d}{dx} \cos^4 x \sin x$
  \[= 4 \cos^3 x (-\sin x) \sin x + \cos^4 x \cos x\]
  \[= \cos^2 x - 4 \cos^3 x \sin^2 x\]

b. $\frac{d}{dx} \frac{ae^{bx}}{cx^2}$
  \[= \frac{a}{c} \left[ x^2 (-b) e^{-bx} - e^{-bx} 2x \right] x^4\]
  \[= -\frac{a}{c} e^{-bx} \left( \frac{b}{x^2} + \frac{2}{x} \right)\]

c. $\frac{d}{dx} 5x^4 = 20x^3$

d. $\frac{d}{dx} x(x^2 - 4)^{1/2}$
  \[= (x^2 - 4)^{1/2} + \frac{x}{2} \left( x^2 - 4 \right)^{1/2} (2x)\]
  \[= (x^2 - 4)^{1/2} + \frac{x^2}{(x^2 - 4)^{1/2}}\]

e. $\sin x' = \cos x$
  $\cos x' = -\sin x$
  $\tan x' = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$
  \[= \frac{1}{\cos^2 x}\]
An integral is a summation process, being the sum of products of the length of an interval and the value of a function at a point in the interval.

By the Fundamental Theorem of Calculus, if \( f \) and \( F \) are continuous on \( a \leq x \leq b \), and \( F'(x) = f(x) \) for \( a \leq x \leq b \), then
\[
\int_a^b f(x) \, dx = F(b) - F(a) = \sum \int_{x_{j-1}}^{x_j} f(x') \, dx'
\]
where \( x' \) is some point between \( x_j \) and \( x_{j-1} \) and \( \Delta x_j = x_j - x_{j-1} \).
An indefinite integral has no bounds, while a definite integral is bounded by continuous functions.

An integral in 2D space represents the area under a graph of the function. In 3D space, the integral represents the volume. Integrals can also be used to find the amount of work associated with a force acting on a moving object.

23. A differential is an incremental value of an independent variable, \( dx \), or function, \( dy = y'dx \).
In general, the derivative is used to represent a rate of change.

A derivative is the limit of a difference-quotient:

\[
 f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]
is the derivative of \( f \) at \( c \).

The 1st derivative represents the slope of a function at a point.
The 2nd derivative represents the curvature of a function at a point.

24. \[
 \frac{Dx^2}{Dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}
= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}
= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x
\]

25. \[
 \lim_{x \to 0} \frac{\sin(x)}{x} : \quad \text{Since } f(x)/g(x) \text{ is indeterminate, use L'Hôpital's Rule,}
\]
\[
 \lim_{x \to 0} \frac{f(x)}{g(x)} = L \quad \text{implies} \quad \lim_{x \to 0} f(x) = L
\]
\[
 \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1
\]

26. Integration by Parts: \( \int udv = uv - \int vdu \)
Chain Rule: \( \frac{d}{dx} f(g(x)) = \left[f'(g(x))\right]g'(x) \)
Quotient Rule: \( \frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \)

27. a. \( f(x) = e^{-x^2} \)
\( f'(x) = -2xe^{-x^2} \)
\( f''(x) = 4x^2e^{-x^2} - 2e^{-x^2} \)
\( (0,1) \) local maximum
b.  \[ f(x) = a \sin x \]
\[ f'(x) = a \cos x \]
\[ f''(x) = -a \sin x \]

\((\pi/2, 1)\) local maximum
\((3\pi/2)\) local minimum
\((\pi, 0)\) inflection point.

c.  \[ f(x) = e^{\frac{\pi x}{2}} \]
\[ f'(x) = 0 \]
\[ f''(x) = 0 \]

no extrema

d.  \[ f(x) = 3x^2 - 17x - 10 \]
\[ f'(x) = 6x - 17 \]
\[ f''(x) = 6 \]

\((17/6, -409/12)\) local minimum

e.  \[ f(x) = x^3 - x^2 \]
\[ f'(x) = x(3x - 2) \]
\[ f''(x) = 6x - 2 \]

\((0,0)\) local maximum
\((2/3, -4/27)\) local minimum
\((1/3, -2/27)\) inflection point

f.  \[ f(x) = x^2 e^{-x} \]
\[ f'(x) = 2 e^{-x} (x - x^3) \]
\[ f''(x) = 2 e^{-x} (1 - 5x^2 + 2x^4) \]

\((\pm1, 1/e)\) local maximum
\((0,0)\) local minimum
\((\pm0.467, 0.177)\) and \((\pm1.51, 0.233)\) are inflection points.
28. y = 1 + e^{-x}
   y' = -e^{-x}  no extrema
   y'' = e^{-x}  no inflection points

29. For a parabola, with vertical axis x=h, vertex (h,k) and focus (h,k+p) the equation in standard form is,
   \[(x - h)^2 = 4p(y - k)\]
   \[y = \frac{(x - h)^2}{4p} + k\]
   \[y' = \frac{2(x - h)}{4p}\]
   \[y'' = \frac{2}{4p}\]

   Extrema are found where y'=0, and the parabola will be concave upward if y'' > 0 or concave downward if y'' < 0. Thus for y'(h)=0, if p > 0 then (h,k) is a minimum, if p < 0 then (h,k) is a maximum.

30. A sphere \(x^2+y^2+z^2 = R^2\) can be represented in the form,
   \[r(u,v) = R \cos v \cos u \hat{i} + R \cos v \sin u \hat{j} + R \sin v \hat{k}\]
   where \(0 \leq u \leq 2\pi\)  \(-\pi/2 \leq v \leq \pi/2\)

   The surface integral is represented by
   \[A(s) = \iiint|\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv\]
   \[= R^2 \iiint|\cos^2 v \cos^2 u + \cos^2 v \sin^2 u + \cos^2 v \sin^2 v|^{1/2} \, dv \, du\]
   \[= R^2 \iiint|\cos^4 v + \cos^4 v - \cos^4 v|^{1/2} \, dv \, du\]
   \[= R^2 \cdot \text{surface area of a sphere.}\]
31. 

a. \( y'' + 6y' + 9 = 5 \)

Try the solution, \( y = e^{\lambda x} \), then the characteristic equation is:

\[
\lambda^2 + 6\lambda = \lambda(\lambda + 6) = 0 \quad \text{with real roots } \lambda = 0, -6
\]

Then a general solution for the homogeneous equation \( (y'' + 6y' = 0) \) is:

\[
y = C_1 e^{-6x} + C_2
\]

Since the given equation is nonhomogeneous, the general solution consists of the general solution of the homogeneous equation plus a particular solution of the nonhomogeneous equation.

Since the right hand side is constant the choice of particular solution is \( y = C_3 x + C_4 \).

Substituting this solution into the differential equation gives

\[
0 + 6C_3 + 9 = 5
\]

\[
C_3 = -\frac{2}{3}
\]

Thus, a general solution to the differential equation is

\[
y = C_1 e^{-6x} - \frac{2}{3} x + C
\]

b. 

\[
\frac{dN}{dt} = -2N
\]

\[
\frac{dN}{N} = -2dt
\]

\[
\int \frac{dN}{N} = -\int 2dt
\]

\[
\ln N = -2t
\]

\[
N = e^{-2t}
\]

c. 

\[
\frac{dy}{dx} = xy^3
\]

\[
\frac{dy}{y^3} = xdx
\]

\[
\int \frac{dy}{y^3} = \int xdx
\]

\[
-\frac{1}{2y^2} = \frac{x^2}{2} + C
\]

\[
y = \frac{1}{\sqrt{2C} - x^2}
\]

Using the initial condition \( x = 0, y = 1 \) then \( C = -1/2 \)

\[
\therefore y = \frac{1}{\sqrt{1 - x^2}}
\]
33. The solution can be reached through trial and error by trying different shapes and differentiating the Area to find the maximum.

**Rectangle:**
\[
\begin{align*}
L + 2W &= 80 \\
A &= L \cdot W \\
A &= (80 - 2W)W \\
dA &= 80 - 4W = 0 \text{ when } W = 20, L = 40 \\
\therefore A_{\text{max, rectangle}} &= 800 \text{ ft}^2
\end{align*}
\]

**Semi-circle:**
\[
\begin{align*}
\pi R &= 80 \\
A &= \frac{\pi R^2}{2} \\
A &= \frac{\pi}{2} (80/\pi)^2 \\
\therefore A_{\text{max, semicircle}} &\approx 1000 \text{ ft}^2
\end{align*}
\]

Thus, a semi-circle inscribes the largest area.
34. \[ V = (L - 2x)(W - 2x)x \]
\[ = LWx - 2x^2W - 2x^2L + 4x^3 \]
\[ V' = LW - 4xW - 4xL + 12x^2 \]
Set \( V' = 0 \) to find maximum or minimum volume
\[ 0 = LW - 4xW - 4xL + 12x^2 = 12x^2 - (4W + 4L)x + LW \]
Solve using the quadratic formula
\[ x = \frac{(4W + 4L) \pm \sqrt{(4W + 4L)^2 - 48LW}}{24} \]
To find the maximum, find when \( V'' < 0 \),
\[ V'' = -4W - 4L + 24x \]
Evaluating using \( x \), a maximum exists for
\[ x = \frac{(4W + 4L) - \sqrt{(4W + 4L)^2 - 48LW}}{24} \]
35. The amount of time the fly spends in the air is the same as the time it takes for the two runners to reach each other. Thus,
\[ t = \frac{d}{v} = \frac{5 \text{ miles}}{5 \text{ mph}} = 1 \text{ hr} \]
\[ d_{fly} = v_{fly} \cdot t = (20 \text{ mph})(1 \text{ hr}) = 20 \text{ miles} \]
36. Laplace Transform: \[ L(s) = \int_{0}^{\infty} e^{-st}f(t)dt \]
Fourier Series: \[ f(x) = \sum_{n=0}^{\infty} a_{n}y_{n}(x) = a_{0}y_{0}(x) + a_{1}y_{1}(x) + \ldots \]
Taylor Series: \[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^{n} \]
37. L'Hopital's Rule is used when evaluating the limit of a function of the form \( q(x) = \frac{f(x)}{g(x)} \) that produces an indeterminate form \( 0/0 \) when \( f(x) \) and \( g(x) \) are continuous and differentiable.
38. Of the 36 possible combinations, 6 of them produce a result of 7. Thus,
\[ p(7) = \frac{6}{36} = \frac{1}{6} \]
39. \[ \frac{dN}{dt} = \lambda N \]
Then, in \( t = 2 \) yrs \( N = 2N_{0} \)
\[ \frac{dN}{N} = \lambda \, dt \]
\[ N = C e^{\lambda t} \]
at \( t = 0 \), \( C = N_{0} \)
\[ \lambda = \frac{\ln 2}{2} \]
\[ N = N_{0} e^{\lambda t} \]
For population \( N = 3N_{0} \)
\[ 2N_{0} = N_{0}e^{2\lambda} \]
\[ 2 = e^{2\lambda} \]
\[ \ln 2 = 2\lambda \]
\[ 3N_{0} = N_{0} e^{\frac{\ln 2}{2}} \]
\[ 3 = e^{\frac{\ln 2}{2}} \]
\[ 2 \ln 3 = \ln 2 \cdot t \]
\[ \therefore t = \frac{2 \ln 3}{\ln 2} = 3.17 \text{ yrs} \]
40. Let \( f(x) = g'(x) \), then (on a unit basis for convenience) the area under the function curve is
\[
\int_0^1 g'(x) \, dx = g(1) - g(0) = 1A
\]
Also, the rectangular area is
\[
f(1) \cdot 1 = g'(1) \cdot 1 = 4A
\]
This is satisfied by \( g(x) = x^4 \)
Thus, \( f(x) = g'(x) = 4x^3 \)

41. The voltage of the power source equals the voltage drop across the resistor and capacitor. Although \( V \), \( R \), and \( C \) are constant, the charge on the capacitor, \( Q \), and the current, \( I \), are functions of time, and \( I = dQ/dt \).
\[
V = IR + \frac{Q}{C}
\]
At \( t = 0 \), \( Q = 0 \), then
\[
\ln\left(1 - \frac{Q}{CV}\right) = -\frac{t}{RC}
\]
Integrating,
\[
\int \frac{dQ}{Q - CV} = -\frac{1}{RC} \int dt
\]
\[
\ln(Q - CV) = -\frac{t}{RC} + K
\]
\[
Q = CV\left(1 - e^{-\frac{t}{RC}}\right)
\]
\[
I = \frac{dQ}{dt} = \frac{V}{R} e^{-\frac{t}{RC}}
\]

42. An example of two equations in two unknown functions \( y_1(t) \) and \( y_2(t) \) are given by
\[
y_1' = a_{11} y_1 + a_{12} y_2
\]
\[
y_2' = a_{21} y_1 + a_{22} y_2
\]
This can be written in Matrix form as,
\[
y' = Ay
\]

\[
y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = Ay = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]
The eigenvalues are found from the trial solution,
\[
y = xe^{\lambda t}
\]
then,
\[
(A - \lambda I)x = 0
\]
solving for the eigenvalues, \( \lambda \), and eigenvectors \( x \), the nontrivial solution of the differential equations becomes,
\[
y^{(1)} = x^{(1)} e^{\lambda_1 t}, \quad y^{(2)} = x^{(2)} e^{\lambda_2 t}, \quad y^{(n)} = x^{(n)} e^{\lambda_n t}
\]

43. \[
\sum_{n=1}^{100} n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \frac{100(101)}{2} = 5050
\]
44. Using the method of composite parts, let $M$ be the total mass, $y$ be the center of mass coordinate, and $\rho$ be the density. Then,

$$M\ddot{y} = m_1\ddot{y}_1 + m_2\ddot{y}_2$$

$$\rho[(1)(1)(a) + (1)(1-a)(a)]\ddot{y} = \rho[(1)(1)(a)]\left[-\frac{a}{2}\right] + \rho[(1)(1-a)(a)]\left[\frac{a}{2}\right]$$

$$[a + a - a^2]\ddot{y} = \left[a^2 - \frac{a^2}{2} - \frac{a^3}{2}\right]$$

$$\therefore \ddot{y} = \frac{a^3}{(4 - 2a)}$$

45. Differential equations can be classified according to three criteria:

**Order:** This is the highest order derivative of the unknown function, $y$ (ex. $y', y'', y'''$ are 1st, 2nd, 3rd order)

**Linearity:** A differential equation is linear in the unknown function $y$ and its derivatives if it contains terms of the form, $y', y'', y'''...$

A nonlinear equation may contain terms such as $y^2, (y'')^3, (y''')^{1/2}...$

**Homogeneity:** If the differential equation = 0, it is homogeneous.

Examples:

- 2nd order, linear, homogeneous equation
  $$y'' + 4y = 0$$

- 2nd order, nonlinear
  $$y'' = \sqrt{y'^2 + 1}$$

46. A general solution of a nonhomogeneous equation is of the form

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ is a general solution, and $x_p(t)$ is a particular solution.

To find the general solution, solve the homogeneous equation. Assume,

$$x_h = Ce^{\lambda t}$$

$$x'_h = C\lambda e^{\lambda t}$$

$$x''_h = C\lambda^2 e^{\lambda t}$$

substituting into the original homogeneous equation gives

$$\lambda^2Ce^{\lambda t} + 5\lambda Ce^{\lambda t} + 6Ce^{\lambda t} = 0$$

$$(\lambda^2 + 5\lambda + 6) = 0$$

$$\lambda = -2, -3$$

Thus, the general solution is

$$x_h(t) = C_1e^{-2t} + C_2e^{-3t}$$
To obtain the particular solution, assume,
\[ x_p = Be^{-t} \]
\[ x'_p = -Be^{-t} \]
\[ x''_p = Be^{-t} \]
substituting into the equation and dividing by \( e^{-t} \) gives,
\[ B - 5B + 6B = 1 \]
\[ B = \frac{1}{2} \]
Thus, the particular solution is
\[ x_p(t) = \frac{1}{2} e^{-t} \]
Finally,
\[ x(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{2} e^{-t} \]

47. \( L(s) = \int e^{-st} f(t) dt = \int e^{-st} t dt \)

Using integration by parts \( u = t, \ du = dt \)
\[ dv = e^{-st} dt \quad v = -\frac{1}{s} e^{-st} \]
\[ = t \left( -\frac{1}{s} e^{-st} \right) \bigg|_0^\infty + \frac{1}{s} \int e^{-st} dt \]
\[ = 0 - \frac{1}{s^2} e^{-st} \bigg|_0^\infty = -\frac{1}{s^2} (0 - 1) = \frac{1}{s^2} \]

48. Assume \( y_h = Ce^{-x} \) then,
\[ \lambda^2 + 4\lambda + 3 = 0 \]
\[ (\lambda + 3)(\lambda + 1) = 0 \quad \text{with roots} -3, -1 \]
Thus, the general solution is
\[ y_h(x) = C_1 e^{-x} + C_2 e^{-3x} \]
To obtain a particular solution assume,
\[ y_p(x) = A \sin(x) + B \cos(x) \]
\[ y'_p(x) = A \cos(x) - B \sin(x) \]
\[ y''_p(x) = -A \sin(x) - B \cos(x) \]
substituting in gives,
\[ -A \sin(x) - B \sin(x) + 4A \cos(x) - 4B \sin(x) + 3A \sin(x) + 3B \cos(x) = \sin(x) \]
\[ (-A - 4B + 3A) \sin(x) + (-B + 4A + 3B) \cos(x) = \sin(x) \]
from the coefficients in front of the sin and cos functions,

\[\begin{align*}
2A - 4B &= 1 \\
4A + 2B &= 0
\end{align*}\]

then, \(A = \frac{1}{10}\) and \(B = -\frac{1}{5}\) and the particular solution becomes

\[y_p(x) = \frac{1}{10}\sin(x) - \frac{1}{5}\cos(x)\]

Finally, the general solution is

\[y(x) = C_1e^{-x} + C_2e^{-3x} + \frac{1}{10}\sin(x) - \frac{1}{5}\cos(x)\]

49.

\[\frac{dx}{dt} = \frac{x}{k}\]

\[\int \frac{dx}{x} = \frac{1}{k} \int dt\]

\[\ln(x) + C = \frac{t}{k}\]

\[\ln(x) = \frac{t}{k} - C\]

\[\therefore x = e^{\frac{t}{k} - C}\]

50. **General Solution**

<table>
<thead>
<tr>
<th>General Solution</th>
<th>Particular Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y' + Ky = 0)</td>
<td>(y' + Ky = 10)</td>
</tr>
<tr>
<td>(y' = -Ky)</td>
<td>Assume (y_p = \frac{10}{K})</td>
</tr>
<tr>
<td>(\frac{dy}{y} = -Kdx)</td>
<td>(y_p = 0)</td>
</tr>
<tr>
<td>(\ln y = -Kx)</td>
<td>(0 + K(10/K) = 10)</td>
</tr>
<tr>
<td>(y_n = e^{-Kx})</td>
<td>(10 = 10), assumption holds</td>
</tr>
</tbody>
</table>

\[\therefore y(x) = e^{-Kx} + \frac{10}{K}\]

51. Assume a solution with undetermined coefficients. Differentiate the trial solution for as many derivatives exist, and substitute into the original equation. Values for the coefficient can be determined by the initial conditions. The solution would be of the form \(A\cos(x) + B\sin(x)\) or exponentials, \(e^\lambda\) where \(\lambda\) is a root of the characteristic equation.

52.

\[y = 3y' = 0\]

\[\frac{dy}{y} = \frac{1}{3} dx\]

Using the initial condition \(y(0) = 3\)

\[\ln y = \frac{1}{3}x + C\]

\[3 = C_1e^0\quad \text{or}\quad C_1 = 3\]

\[y = e^{\frac{1}{3}x} = e^{-e^{\frac{1}{3}x}} = C_1e^{\frac{1}{3}x}\]

\[\therefore y(x) = 3e^{\frac{x}{3}}\]
1. In the y-direction, the y component of the gravity force balances the normal force, and in the x-direction, the force of friction balances the x component of the gravity force just before the block starts sliding.

\[ \sum F_y = 0 = N - F_{gy} \]
\[ = N - Mg \cos \theta \]
\[ \sum F_x = 0 = F_{gx} - F_{fr} \]
\[ = Mgsin \theta - \mu N \]

substituting in for the normal force,
\[ \sum F_x = Mgsin \theta - \mu Mgsin \theta \]
\[ \theta = \tan^{-1}(\mu) = \tan^{-1}(0.8) \]
\[ \theta = 38.66^\circ \]

2. conservation of momentum:
\[ m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \]
\[ MV = MV' + mv' \]

conservation of energy:
\[ \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \]
\[ \rightarrow \frac{1}{2}MV_0^2 = \frac{1}{2}MV'^2 + \frac{1}{2}mv'^2 \]

In an elastic collision,
\[ v_1 - v_2 = v_2' - v_1' \]
\[ \rightarrow V_0 = v - V \]

thus,
\[ v_2' = v_1 \left( \frac{2m_2}{m_1 + m_2} \right) \]
\[ v' = v_0 \left( \frac{2M}{M + m} \right) \]
\[ v_1' = v_1 \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \]
\[ v' = v_0 \left( \frac{M - m}{M + m} \right) \]

Assuming a completely inelastic collision
\[ MV_0 = (M + m)V' \]
\[ v' = \frac{V_0M}{(M + m)} \]

3. The spring-mass system will undergo simple harmonic motion
\[ F_s = -kx \]
\[ M \frac{d^2x}{dt^2} = -kx \]
\[ \frac{d^2x}{dt^2} + \frac{k}{M} x = 0 \]

let \( \omega^2 = \frac{k}{M} \]
\[ 4 = a \cos \omega t + b \sin \omega t \]
\[ 0 = 4 \sin \omega t + b \cos \omega t \]
\[ a = 4 \]
\[ b = 0 \]

The general solution is
\[ x = a \cos \omega t + b \sin \omega t \]
\[ x = 4 \cos \omega t \]
4. The gravitational force and the electrostatic force both depend upon the inverse of the distance squared,
\[ F_{\text{grav}} = G \frac{m_1 m_2}{r^2}, \quad F_{\text{elec}} = k \frac{q_1 q_2}{r^2} \]
Thus, if the distance is doubled, the Force between two masses or two charged particles will be \( \frac{1}{4} \) the original.

5. \[ d_x = V_{ox} \cdot t \]
\[ d_y = V_{oy} + \frac{1}{2} at^2 = \frac{1}{2} gt^2 \]
\[ 555 \text{ ft} = \frac{1}{2} (32.2 \text{ ft/s}) t^2 \]
\[ t = 5.87 \text{ s} \]
\[ d_x = (50 \text{ ft/s})(5.87 \text{ s}) \]
\[ d_x = 293.5 \text{ ft} \]

6. Newton's second law, can be stated in terms of momentum, \( p = Mv \), as
\[ F_{\text{ext}} = \frac{dp}{dt} = M \frac{dv}{dt} + v \frac{dM}{dt} \]
For a rocket ship, expending fuel at a rate \( dM/dt \), the velocity of \( dM \) is the relative velocity with respect to \( M \)
\[ v_{\text{rel}} = u_{\text{thrusts}} - v_{\text{rocket}} \]
If the spaceship is in outer space sufficiently far away from any large masses, then the external force due to gravity is negligible. The term \( v_{\text{rel}} dM/dt \), referred to as the thrust and is the force exerted on the spaceship by the expelled gas. Thus,
\[ M \frac{dv}{dt} = v_{\text{rel}} \frac{dM}{dt} \]
\[ \frac{dv}{u - v} = \frac{dM}{M} \]
For the spaceship to come to rest from initial velocity \( V_0 \),
\[ \int_0^{V_0} \frac{dv}{u - v} = \int_{M_0}^{M} \frac{dM}{M} \]
\[ -\ln(u - v)_{V_0}^u = \ln M_{M_0}^M \]
\[ \ln \left( \frac{u - V_0}{u} \right) = \ln \frac{M_0}{M} \]
\[ \frac{u - V_0}{u} = \frac{M_0}{M} \]
\[ \therefore u = \frac{V_0}{1 - \frac{M_0}{M}} \]
Since the final mass, \( M \), will be less than the initial mass, \( M_0 \), the velocity will be negative, indicating a reverse thrust.
To find the final mass requires knowledge of the time it takes to stop the spaceship from the initial velocity and knowledge of the fuel consumption rate, \( dM/dt \).
7. This problem can be broken down into a number of smaller problems and solved using conservation of energy principles, assuming no losses. For the car of mass, \( M \), starting at height, \( h \), the velocity at the bottom of the decline, \( V_1 \), can be found by converting the potential energy of the car into kinetic energy

\[
\frac{1}{2} MV_1^2 = Mgh
\]

\[
V_1 = \sqrt{2gh}
\]

As the car goes around the loop, if the force of gravity overcomes the centrifugal force, the car will fall off the track. The minimum velocity of the car will occur at the highest point on the loop, since some of the kinetic energy of the car has been converted back into potential energy. Thus,

\[
\sum F_y = 0
\]

\[
M \frac{V_2^2}{r} = Mg
\]

\[
V_{2,min} = \sqrt{gr}
\]

where \( V_2 \) is the minimum velocity allowed to keep the car from falling off the track. From conservation of energy

\[
\Delta KE = \Delta PE
\]

\[
\frac{1}{2} MV_i^2 - \frac{1}{2} MV_2^2 = Mg(2r)
\]

\[
V_1 = \sqrt{2gh}
\]

\[
\frac{1}{2} M(2gh) - \frac{1}{2} MV_2^2 = Mg(2r)
\]

\[
\frac{1}{2} V_2^2 = g(h - 2r)
\]

\[
V_2 = \sqrt{2g(h - 2r)}
\]

To solve for \( h \), equate \( V_{2,min} \) to \( V_2 \) to get

\[
\sqrt{2g(h - 2r)} = \sqrt{rg}
\]

\[
2g(h - 2r) = rg
\]

\[
\therefore h = \frac{5}{2}r
\]

With no losses, the velocity after the loop will be \( V_1 \) since all the potential energy of the loop has been converted back into the initial kinetic energy. Thus, from the spring-mass equations the displacement, \( x \), can be found assuming all the kinetic energy of the car is converted into potential energy of the spring

\[
PE_s = \frac{1}{2} kx^2 = \frac{1}{2} MV_1^2
\]

\[
\therefore x = \sqrt{\frac{MV_1^2}{k}}
\]
8. \[ x = x_0 + v_0 \cos \theta t + \frac{1}{2} a_t t^2, \quad x_0 = 0, \quad a_x = 0 \]
\[ y = y_0 + v_0 \sin \theta t + \frac{1}{2} a_y t^2, \quad y_0 = 0, \quad a_y = -g \]

Solving for the time, \( t \), it takes until the projectile hits the ground (\( y=0 \)) gives
\[ 0 = v_0 \sin \theta t - \frac{1}{2} gt^2 \]
\[ t = \frac{2v_0 \sin \theta}{g} \]

Then the distance the projectile travels can be given as
\[ x = v_0 \cos \theta t \]
\[ = v_0 \cos \theta \frac{2v_0 \sin \theta}{g} \]
\[ = \frac{v_0^2}{g} 2 \sin \theta \cos \theta = \frac{v_0^2}{g} \sin 2\theta \]

This is a maximum when \( \sin 2\theta = 1 \), or \( \theta = 45 \) degrees.

9. Surface tension will prevent the water from spilling out. The surface is unperturbed and will remain unperturbed as long as the paper is there. The water is held together by the strong cohesive forces between water molecules, which is very strong with a tensile strength of \( 30 \times 10^6 \) N/m\(^2\). However, the slightest disturbance and the water will begin to flow under the influence of gravity.

10. \[ \Delta KE_{\text{translational}} + \Delta KE_{\text{rotational}} = \Delta PE, \quad \text{initial KEs = 0} \]
\[ \frac{1}{2} Mv_s^2 + \frac{1}{2} I_{cm} \omega^2 = Mgy, \quad \omega = \left( \frac{v}{R} \right), \quad y = h \]

The cylinders have equal outer radius, \( R_0 \) and the hollow cylinder has inner radius \( R_i \). Thus,

For the solid cylinder:
\[ I_{cm} = \frac{1}{2} MR_0^2 \]
\[ \frac{1}{2} Mv_s^2 + \frac{1}{2} \left( \frac{1}{2} MR_0^2 \right) \left( \frac{v_s}{R_0} \right)^2 = Mgh \]
\[ \left( \frac{1}{2} + \frac{1}{4} \right) v_s^2 = gh \]
\[ v_s = \sqrt{\frac{gh}{\left( \frac{1}{2} + \frac{1}{4} \right)}} \]

For the hollow cylinder:
\[ I_{cm} = \frac{1}{2} MR_0^2 + \frac{1}{2} MR_i^2 \]
\[ \frac{1}{2} Mv_h^2 + \frac{1}{2} \left( \frac{1}{2} M(R_0^2 + R_i^2) \right) \left( \frac{v_h}{R_0} \right)^2 = Mgh \]
\[ \left( \frac{1}{2} + \frac{1}{4} \right) \frac{R_0^2}{R_i^2} v_h^2 = gh \]
\[ v_h = \sqrt{\frac{gh}{\left( \frac{1}{2} + \frac{1}{4} \right) \frac{R_0^2}{R_i^2}}} \]

\( v_s > v_h \) therefore, the solid cylinder will reach the bottom first.
11. For the TV with electric field strength, \( E \), perpendicular to the initial velocity.

\[
a_y = \frac{F}{m_e} = -\frac{qE}{m_e}
\]

\[
y = \frac{1}{2} a_y t^2 = -\frac{qE}{2m_e} t^2
\]

\[
x = v_0 t \quad \rightarrow \quad t = \frac{x}{v_0}
\]

\[
:\therefore y = -\frac{qE}{2m_e v_0^2} x^2
\]

The electron follows the path of a parabola. Where it hits the screen can be changed by altering \( E \) and \( v_0 \).

12. \[
y = y_0 + v_0 \sin \theta t + \frac{1}{2} at^2
\]

\[
x = v_0 \cos \theta t
\]

\[
v_0 \sin \theta (4) = \frac{1}{2} g(16)
\]

\[
300 = \frac{19.6}{\sin \theta} \cos \theta (4)
\]

\[
\therefore v_0 = \frac{19.6}{\sin \theta}
\]

\[
\therefore \theta = 14.646
\]

The projectile reaches maximum height at time \( t=2 \) seconds

\[
y = y_0 + v_0 \sin \theta t + \frac{1}{2} at^2
\]

\[
y = 0 + \frac{19.6}{\sin(14.646)} (2) + \frac{1}{2} g(2)^2
\]

\[
y = 19.6 \text{ ft}
\]

13. 1st Law: A body continues in its state of rest or of uniform speed in a straight line unless it is compelled to change that state by forces acting on it. This law is also known as the law of inertia.

2nd Law: The acceleration of an object is directly proportional to the net force acting on it and is inversely proportional to its mass. The direction of the acceleration is in the direction of the applied next force. \( \mathbf{F} = m \mathbf{a} \).

3rd Law: Whenever one object exerts a force on a second object, the second exerts an equal and opposite force on the first. More commonly stated as, "For every action, there is an equal and opposite reaction."

14. Assuming an inelastic collision

\[
m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_r
\]

\[(10 \text{ g})(1000 \text{ m/s}) = (1010 \text{ g}) v_r \]

\[
v_r = 9.9 \text{ m/s}
\]

The change in kinetic energy will equal the rise in potential energy

\[
\frac{1}{2} (m_1 + m_2) v_r^2 = (m_1 + m_2)gh
\]

\[
h = \frac{v_r^2}{2g}
\]

\[
\therefore h = 5 \text{ m}
\]
15. \[ x = x_0 + v_o \cos \theta t \]
\[ y = y_0 + v_o \sin \theta t - \frac{1}{2} gt^2, \quad y_0 = 0 \]
\[ v_o \sin \theta t = \frac{1}{2} gt^2 \]
\[ t = \frac{2v_o \sin \theta}{g} \]
\[ \therefore x = \frac{2v_o^2}{g} \sin \theta \cos \theta = \frac{v_o^2}{g} \sin 2\theta \]

The distance the rocket sled travels depends on the initial velocity and the angle of the ramp.

16. \[ y = y_0 + v_{oy} t - \frac{1}{2} gt^2, \quad y_0 = h, \quad v_{oy} = 0 \]
\[ h = \frac{1}{2} gt^2 \]
\[ \therefore t = \frac{\sqrt{2h}}{g} \]

17. \[ x_{man} = x_0 + v_{0,man} t + \frac{1}{2} a_{man} t^2 = x_0 + v_{0,man} t \]
\[ x_{bus} = (x_0 + z) + v_{0,bus} t + \frac{1}{2} a_{bus} t^2 = (x_0 + z) + \frac{1}{2} a_{bus} t^2 \]

If \( x_{man} > x_{bus} \) then he will catch the bus. This will occur if the separation distance \( z \) is less than a certain distance given by

\[ x_0 + v_{0,man} t > (x_0 + z) + \frac{1}{2} a_{bus} t^2 \]
\[ z < 3t - \frac{1}{2} t^2 \]

18. When the block is released, the potential energy is changed into kinetic energy, and some of that energy will be transferred to the stationary block.
\[ \frac{1}{2} M_1 v_1^2 = M_1 gh \]
\[ v_1 = \sqrt{2gh} \]

For a totally elastic collision
\[ M_1 v_1 + M_2 v_2 = M_1 v_1' + M_2 v_2' \]
\[ \frac{1}{2} M_1 v_1^2 + \frac{1}{2} M_2 v_2^2 = \frac{1}{2} M_1 v_1'^2 + \frac{1}{2} M_2 v_2'^2 \]
\[ \therefore v_2' = v_1 \left( \frac{2M_1}{M_1 + M_2} \right) = \sqrt{2gh} \left( \frac{2M_1}{M_1 + M_2} \right) \]
19. Due to the laws of conservation of energy and momentum, the first ball transfers its momentum and energy to the inner spheres, each one transferring its energy to the next sphere. Because no sphere is located next to the end ball, it retains the momentum and energy and swings outward.

20. Using conservation of momentum, assuming an inelastic collision

\[ M_1v_1 = (M_1 + M_2)v_f \]
\[ v_f = \frac{M_1v_1}{(M_1 + M_2)} = \frac{(10 \text{ g})(1000 \text{ m/s})}{(110 \text{ g})} = 90.9 \text{ m/s} \]

This problem cannot be solved using conservation of energy since the collision is inelastic and some of the initial kinetic energy has been transformed into other types such as thermal. This can be seen by comparing the kinetic energy before and after the collision

\[ \frac{1}{2}M_1v_1^2 = \frac{(0.01 \text{ kg})(1000 \text{ m/s})^2}{2} = 5000 \text{ J} \]
\[ \frac{1}{2}(M_1 + M_2)v_f^2 = \frac{(0.110 \text{ kg})(90.9 \text{ m/s})^2}{2} = 455 \text{ J} \]

Thus, about 4545 J of energy was converted to other forms. Since the specific heat of wood is 1700 J/kg·C, if all the residual energy was transformed into thermal energy, the temperature of the wood block would have been raised by 24°C.

21. Momentum is the product of the mass of a body and its velocity, \( p=mv \). Newton's Second Law states, Force = mass x acceleration

\[ \ddot{F} = ma = m \frac{dv}{dt} \]
\[ p = mv, \quad \frac{dp}{dt} = m \frac{dv}{dt} \]
\[ \therefore F = \frac{dp}{dt} \]

22. \( y = y_0 + v_{oy}t - \frac{1}{2}gt^2, \quad y_0 = 0, \quad v_{oy} = 100 \text{ ft/s} \)

To find the time at maximum height, differentiate and set \( y' = 0 \).

\[ y' = v_{oy} - gt \]
\[ t = \frac{v_{oy}}{g} = \frac{100 \text{ m/s}}{32.2 \text{ m/s}^2} = 3.106 \text{ s} \]
\[ y = 100(3.106) - \frac{1}{2}(32.2)(3.106)^2 \]
\[ \therefore y = 155.3 \text{ ft} \]

23. The potential energy of the mass has been almost all converted into kinetic energy just before it strikes the ground. Thus,

\[ mgh = \frac{1}{2}mv^2 \quad \rightarrow \quad v = \sqrt{2gh} \]
24. a. \[ mgh = \frac{1}{2}mv_{\text{max}}^2 \rightarrow v_{\text{max}} = \sqrt{2gh} \]

b. Assumptions made are, no external, non-conservative forces act on the system. The string is weightless and in-extensible.

c. The difference is the loss of air friction, a negligible effect.

d. For an elastic collision, by conservation of momentum and kinetic energy,

\[ v'_1 = v_i \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \quad v'_2 = v_i \left( \frac{2m_1}{m_1 + m_2} \right) \]

e. If the collision is non-elastic

\[ v' = \sqrt{2gh} \left( \frac{m_1}{m_1 + m_2} \right) \]

25. Work is the product of force-distance. The amount of work required to move the block 3 units is

\[ W = F \cdot d \]

\[ = \int_3^0 F \, dx \]

\[ = \int_0^3 e^x \, dx \]

\[ = e^3 - e^0 \]

\[ = e^3 - 1 \]

26. Work is defined to be the product of the magnitude of the displacement times the component of the force parallel to the displacement. Energy is the ability to do work. Power is the rate at which energy is transformed.